Lecture 21:
• Quiz 5
• Wrap up Edmonds-Karp Max-Flow Algorithm
• Cliques

Announcements
• HW-5 out (due Tues, 11/15)
• PS-6 out (soon)
• Project Step 2 out (due Thurs 11/17)

Cost of Edmonds-Karp

(1) Assume the cost to compute the residual network is \( O(E) \)
• iterate over edges in \( G \) and the flow \( f \)

(2) Cost to find an augmenting path (via BFS shortest paths) is \( O(E) \)
• recall each node lies on a path in \( G \)
• the residual network contains \( |E| \) to \( 2|E| \) edges ... backflow edges
• augmenting paths are computed from residual networks
• while BFS is \( O(V + E) \), each node involved in at least one edge
• thus \( |V| \) is \( O(E) \)

(3) There are at most \( O(V E) \) augmenting paths ... while loop iterations
• since each iteration finds \( G_f \) and an augmenting path ... with cost \( O(E) \)
• total cost of Edmonds-Karp is \( O(V E^2) \)
Cost of Edmonds-Karp

Why at most $O(V E)$ augmenting paths?

Edge $(u, v)$ is critical on augmenting path $p$ if $c_f(p) = c_f(u, v)$ (the min)

- every augmenting path has at least one critical edge
- the critical edge is removed in the next residual network
- and a new “reversed” edge is added

(a) Assume $(u, v)$ is a critical edge, and $u$ is at BFS level $i$

\[ \begin{align*}
  &i & &i + 1 \\
  &p_1 & & \\
  &s &\rightarrow &u &\rightarrow &v &\rightarrow &t \\
  &s &\rightarrow &u &\rightarrow &v &\rightarrow &t
\end{align*} \]

Note that $p_1$ is a shortest path to $u$ (along augmenting path $p$)

(b) Assume later $(v, u)$ becomes a critical edge ...

\[ \begin{align*}
  &i + 2 & &i + 1 \\
  &u &\leftarrow &v &\rightarrow &t \\
  &s &\rightarrow &u &\rightarrow &v &\rightarrow &t
\end{align*} \]

A shorter path $s \sim u$ (like $p_1$) implies $s \sim v \rightarrow u \sim t$ isn’t a shortest path

- thus, from when $(u, v)$ becomes critical to the next time it becomes critical
- $u$’s distance from $s$ has increased by at least 2 levels!

An edge $(u, v)$ can become critical at most $|V|/2$ times

- since $u$ “jumps” 2 levels at a time and there are at most $|V|$ levels

Since there are $E$ total edges, there are at most $O(V E)$ total critical edges

- and each augmenting path has at least one critical edge
In a **complete** graph, every pair of vertices are connected

- In an undirected graph, an edge \(\{u, v\}\) between each pair of vertices \(u, v\)
- In a directed graph, edges \((u, v)\) and \((v, u)\) for each pair \(u, v\)

A **clique** is a maximal complete subgraph of a graph

- Thus, each pair of vertices in the clique are connected
- And we can’t add a vertex to the graph and maintain completeness

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**Check In:** For the graph give ...
- The maximal cliques \(\{0, 1\}, \{1, 2\}, \{1, 4\}, \{4, 5\}, \{2, 3, 5\}\)
- The maximum clique(s) \(\{2, 3, 5\}\)
- The non-maximum clique(s) \(\{2, 3\}, \{2, 5\}, \{3, 5\}\)

**Check In:** A \(k\)-clique contains \(k\) nodes. Draw a 4-clique and a 5-clique.
A graph can (clearly) have many cliques
• The largest clique in a graph is a maximum clique

From graph theory: at most an exponential number of cliques in a graph
• See Moon and Moser, “On cliques in graphs” (1965)
• e.g., at most $3^{n/3}$ maximal cliques for $n \geq 3$ and $0 = n \mod 3$

Finding the maximum clique is another NP-Complete problem
• pick the largest in an exponential set of cliques

For HW-6, we’ll find all the (maximal) cliques in an (undirected) graph
• by definition a clique is a maximal connected subgraph